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http://smartURL.it/krzysioP (paper) or http://smartURL.it/krzysioPe (electronic)
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Dr. Ostaszewski's online exercise posted May 5, 2007
at: http://smarturl.it/5-5-7-P.pdf,
video posted at: http://smartURL.it/5-5-7-P

You are given that the random variable X is exponential with mean 1, and that the random variable Y is uniformly distributed on the interval $[0,1]$. Furthermore, it is known that X and Y are independent. Find the density of the joint distribution of $U = XY$ and $V = \frac{X}{Y}$.

A. $\frac{2e^{-\sqrt{uv}}}{v}$ for $u > 0$, $v > 0$, and $u < v$
B. $\frac{e^{-\sqrt{uv}}}{2v}$ for $u > 0$, $v > 0$, and $u < v$
C. $2ve^{\sqrt{uv}}$ for $u > 0$ and $v > 0$
D. $2ve^{\sqrt{uv}}$ for $u > 0$, $v > 0$, and $u < v$
E. $\sqrt{uv} \cdot e^{\sqrt{uv}}$ for $u > 0$, $v > 0$, and $u < v$

Solution.
We have $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = e^{-x} \cdot 1 = e^{-x}$ for $x > 0$ and $0 < y < 1$. In order to find the joint density of U and V, we need to first express X and Y in terms of U and V, i.e., find the inverse function of the transformation. Note that all variables considered here: X, Y, U, V, are positive with probability one. We have

$$UV = XY \cdot \frac{X}{Y} = X^2,$$

so that $X = \sqrt{UV}$. Furthermore,

$$\frac{U}{V} = \frac{XY}{XY^{-1}} = Y^2.$$

This gives us $Y = \sqrt{\frac{U}{V}}$. Therefore, the inverse transformation, written in terms of regular variables, is

$$(x, y) = \left(\sqrt{uv}, \sqrt{\frac{u}{v}} \right).$$

This results in

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2\sqrt{uv}} \cdot v & \frac{1}{2\sqrt{uv}} \cdot u \\ \frac{1}{2\sqrt{uv^{-1}}} \cdot v^{-1} & -\frac{1}{2\sqrt{uv^{-1}}} \cdot uv^{-2} \end{bmatrix} = -\frac{1}{2v}.$$

Therefore

$$f_{U,V}(u,v) = f_{X,Y}\left(x(u,v), y(u,v)\right) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{e^{-\sqrt{uv}}}{2v}$$

for $\sqrt{uv} > 0$ and $0 < \sqrt{uv^{-1}} < 1$, i.e., $u > 0$, $v > 0$, and $u < v$.
Answer B.

An interesting additional question was brought up by a student in relation to this problem. If we want to find the marginal density of $U = XY$, then we can do the calculation

$$f_U(u) = \int_u^{+\infty} \frac{e^{-\sqrt{uv}}}{2v} dv = \left| \begin{array}{ll} x = \sqrt{uv} & v = u \Rightarrow x = u \\ \frac{dx}{dv} = \frac{u}{2\sqrt{uv}} & v \to \infty \Rightarrow x \to \infty \end{array} \right| =$$

$$= \int_u^{+\infty} \frac{e^{-x}}{2v} \cdot \frac{dx \cdot 2\sqrt{uv}}{u} = \int_u^{+\infty} \frac{e^{-x}}{x} dx,$$

for $u > 0$, and for the calculation of the marginal density of $V = \frac{X}{Y}$ we have

$$f_V(v) = \int_0^v \frac{e^{-\sqrt{uv}}}{2v} du = \left| \begin{array}{ll} z = \sqrt{uv} & u = 0 \Rightarrow z = 0 \\ \frac{dz}{du} = \frac{1}{2\sqrt{uv}} \cdot v & u = v \Rightarrow z = v \end{array} \right| =$$

$$= \int_0^v \frac{e^{-z}}{2v} \frac{dz}{v} \cdot z = \frac{1}{2v^2} \int_0^v ze^{-z}\, dz = \left| \begin{array}{ll} s = z & t = -e^{-z} \\ ds = dz & dt = e^{-z} dz \end{array} \right| =$$

$$= \frac{1}{2v^2} \left(-ze^{-z} \Big|_{z=0}^{z=v} + \int_0^v e^{-z}\, dz \right) = \frac{1}{2v^2}\left(-ve^{-v} + 0 + 1 - e^{-v} \right) = \frac{1}{2v^2}\left(1 - e^{-v} - ve^{-v} \right)$$

for $v > 0$. These two results look pretty complicated and it is tempting to ask the following question inspired by intuition given by convolution: If the probability density function of the distribution of the sum of independent random variables X and Y is established by the formula

$$f_{X+Y}(s)=\int_{-\infty}^{+\infty} f_X(x) f_Y(s-x)\,dx$$

can't we calculate the probability density function of the distribution of the product of independent random variables X and Y as

$$f_{X\cdot Y}(s)=\int_{-\infty}^{+\infty} f_X(x) f_Y\left(\frac{s}{x}\right)dx?$$

Well, no. Sometimes intuition can be misleading. We need to understand first where the formula

$$f_{X+Y}(s)=\int_{-\infty}^{+\infty} f_X(x) f_Y(s-x)\,dx$$

comes from. Let us start with the formula given by the definition of the cumulative distribution function

$$F_{X+Y}(s)=\Pr(X+Y\le s)=\int_{-\infty}^{+\infty}\int_{-\infty}^{s-x} f_Y(y) f_X(x)\,dy\,dx=$$

$$=\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{s-x} f_Y(y)\,dy\right) f_X(x)\,dx=\int_{-\infty}^{+\infty} F_Y(s-x) f_X(x)\,dx.$$

Therefore,

$$f_{X+Y}(s)=\frac{d}{ds}F_{X+Y}(s)=\frac{d}{ds}\int_{-\infty}^{+\infty} F_Y(s-x) f_X(x)\,dx=\int_{-\infty}^{+\infty}\frac{d}{ds}F_Y(s-x) f_X(x)\,dx=$$

$$=\int_{-\infty}^{+\infty}\underbrace{f_Y(s-x)\cdot 1}_{=\frac{d}{ds}F_Y(s-x)}\cdot f_X(x)\,dx=\int_{-\infty}^{+\infty} f_Y(s-x) f_X(x)\,dx.$$

And there we have the well-known to you (and if it is not well-known to you, it should be on the day when you take the exam) convolution formula for the density of a sum of two independent random variables. But now let us apply the reasoning carefully to a product of two independent random variables, with the product of two random variables called $U=XY$, as we called it in the above exercise. We have

$$F_{X\cdot Y}(u)=\Pr(X\cdot Y\le u)=\int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{u}{x}} f_Y(y) f_X(x)\,dy\,dx=$$

$$=\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{\frac{u}{x}} f_Y(y)\,dy\right) f_X(x)\,dx=\int_{-\infty}^{+\infty} F_Y\left(\frac{u}{x}\right) f_X(x)\,dx,$$

and

$$f_{X\cdot Y}(u)=\frac{d}{du}F_{X\cdot Y}(u)=\frac{d}{du}\int_{-\infty}^{+\infty}F_Y\left(\frac{u}{x}\right)f_X(x)\,dx=\int_{-\infty}^{+\infty}\frac{d}{du}F_Y\left(\frac{u}{x}\right)f_X(x)\,dx=$$

$$=\int_{-\infty}^{+\infty}\underbrace{f_Y\left(\frac{u}{x}\right)\cdot\frac{1}{x}}_{=\frac{d}{du}F_Y\left(\frac{u}{x}\right)}\cdot f_X(x)\,dx=\int_{-\infty}^{+\infty}\frac{1}{x}f_Y\left(\frac{u}{x}\right)f_X(x)\,dx.$$

And this is the formula for the density of a product of two independent random variables. This is a fair warning: In mathematics, intuition can be misleading. You always need to check if intuition is leading you astray. Here is some information on probably the most fascinating example of intuition being knocked out and left unconscious by mathematics, the Banach-Tarski Paradox: https://en.wikipedia.org/wiki/Banach–Tarski_paradox.